Analysis I Lecture 15.

Last time

Limit of functions:

$$\lim_{x \to x_0} f(x) = \ell \iff \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t.}$$

for any x with

$0 < |x - x_0| < \delta$ we have

$$|f(x) - \ell| < \varepsilon$$

- Alternatively:

$$\lim_{x \to x_0} f(x) = \ell \iff \forall \text{ sequence } (a_n) \text{ with}$$

$$a_n \neq x_0 \quad \lim_{n \to \infty} a_n = x_0$$

$$\text{we have } \lim_{n \to \infty} f(a_n) = \ell.$$

Uniqueness of limit.

Proposition $\lim_{x \to x_0} f(x)$ is unique if it exists.

We can use it to show that $\lim_{x \to x_0} f(x)$ do not exists!

Need to find (a_n), (b_n) s.t.

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = x_0, \text{ but } \lim_{n \to \infty} f(a_n) \neq \lim_{n \to \infty} f(b_n)$$

Today:

- Continuous functions
- Algebra of limits
- Infinite limits.
- compositions of functions with connection to limits.

Continuous functions

We saw examples where

- $\lim_{x \to x_0} f(x)$ does not exists

  e.g $\chi_{\mathbb{Q}}(x)$

  we showed that $\lim_{x \to 0} \chi_{\mathbb{Q}}(x)$ does not exists.

- $\lim_{x \to x_0} f(x)$ exists but $\neq f(x_0)$

  e.g. $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

  $\lim_{x \to 0} f(x) = 0 \neq f(0) = 1.$

Definition. Let $f: D \to \mathbb{R}$ be a function. We say that f is continuous at $x_0 \in D$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

In particular, $\lim_{x \to x_0} f(x)$ exists.

- We say that f is continuous if it is continuous at every point $x_0 \in D$.

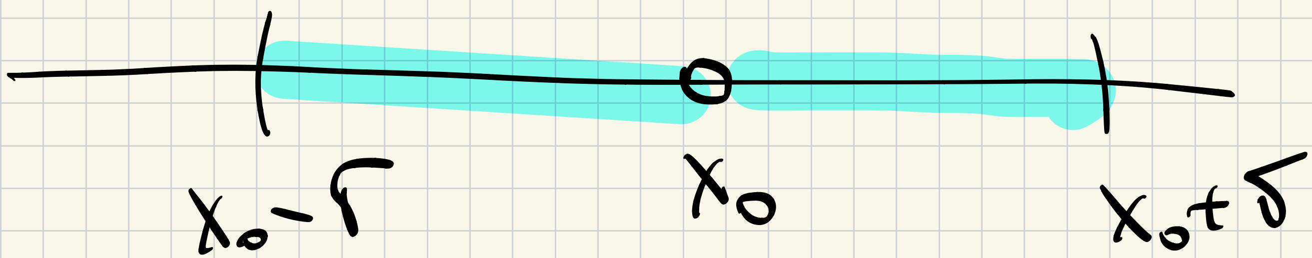
Remark For definition of the limit at $x_0 \in D$ we need to assume that D contains some punctured neighborhood of x_0

Some terminology We can also say that f is continuous on $U \subset D$. That means f is continuous for every $x \in U$.

Punctured neighborhood of x_0

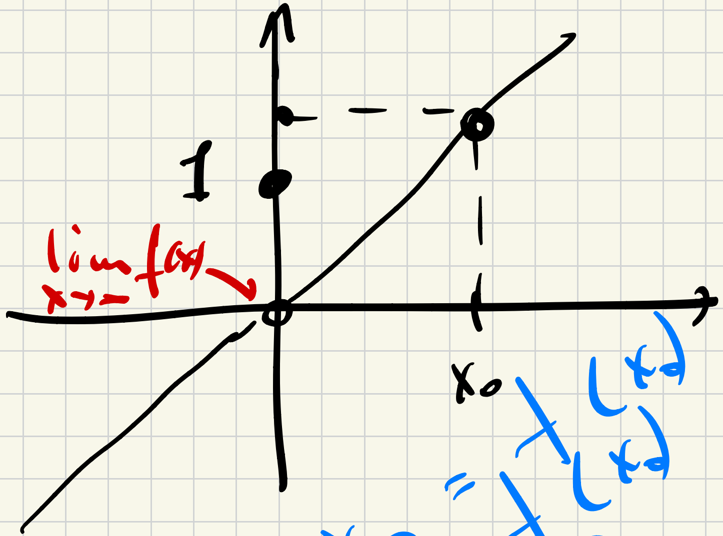
is a set:

$$(x_0 - \delta, \; x_0 + \delta) \setminus \{x_0\}$$



Examples

- $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



$\lim_{x \to x_0} f(x) = x_0 = f(x_0)$ if $x_0 \neq 0$.

f is continuous for every $x \in \mathbb{R}^*$

but discontinuous at 0.

$\lim_{x \to 0} f(x) = 0 \neq 1 = f(0)$

Let's proove that $\lim\limits_{x \to 0} f(x) = 0$

Let (a_n) be a sequence s.t $a_n \neq 0$

but $\lim\limits_{n \to \infty} a_n = 0$.

Need to show that $\lim\limits_{n \to \infty} f(a_n) = 0$

But $f(a_n) = a_n \Rightarrow \lim\limits_{n \to \infty} f(a_n) = \lim\limits_{n \to \infty} a_n = 0$ ■

- $\chi_{\mathbb{Q}}(x)$

Exercise

$\chi_{\mathbb{Q}}$ is discontinuous at 0.

Because $\lim_{x \to 0} \chi_{\mathbb{Q}}(x)$ does not exists.

In fact, the same argument can be used to show that $\chi_{\mathbb{Q}}(x)$ is discontinuous at every $x \in \mathbb{R}$!

- $\cos(x)$

Today $\cos(x)$ is continuous at 0.

(But in fact it is continuous for every $x \in \mathbb{R}$.)

Need to show that $\lim_{x \to 0} \cos(x) = \cos(0) = 1$

Let's prove that $\lim_{x \to 0} \cos(x) = 1$:

Use second definition:

Let a_n be a sequence s.t. $\lim_{n \to +\infty} a_n = 0$

<u>then</u>

$$0 \leq |\cos(a_n) - 1| = \left|2\sin^2\left(\frac{a_n}{2}\right)\right| \leq$$

by $|\sin(x)| < |x|$ $$\leq 2\left(\frac{a_n}{2}\right)^2 = \frac{a_n^2}{2}$$

So we get that

$$0 < \left|\cos(a_n) - 1\right| \leq \frac{a_n^2}{2}$$

0 as $n \to \infty$ $\qquad$ 0 as $n \to \infty$

$\Rightarrow$

by squeeze thm: $\displaystyle\lim_{n\to\infty} \left|\cos(a_n) - 1\right| = 0$

$$\Rightarrow \lim_{n\to\infty} \cos(a_n) = 1.$$

$\blacksquare$

Algebra of limits

Proposition (= Prop 1.20 in Notes on Functions).

Let $D, E \subset \mathbb{R}$, $x_0, a, b \in \mathbb{R}$, $f: D \to \mathbb{R}$, $g: E \to \mathbb{R}$ s.t.

$$\lim_{x \to x_0} f(x) = a \quad ; \quad \lim_{x \to x_0} g(x) = b$$

then

1) $$\lim_{x \to x_0} (f+g)(x) = a + b$$

2) $$\lim_{x \to x_0} (f \cdot g)(x) = a \cdot b$$

3) If $b \neq 0$ then $$\lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \frac{a}{b}$$

<u>Proposition</u> (continuation)

(4) If $\exists\, \delta$ s.t. $f(x) = g(x)$ for every $0 < |x - x_0| < \delta$, then

$$\lim_{x \to x_0} f = \lim_{x \to x_0} g$$

(5) If $\exists\, \delta$ s.t. $f(x) \leq g(x)$ for every $0 < |x - x_0| < \delta$, then

$$\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} g$$

Proof idea:

Use algebra of limits for sequences:

By definition 2: $\lim_{x \to x_0} (f+g) = \ell$ if

for any $(a_n) \to x_0$ as $n \to \infty$

$$\lim_{n \to \infty} \boxed{(f+g)(a_n)} = \ell$$

$= f(a_n) + g(a_n)$

So if $\lim_{n\to\infty} f(a_n) = a$

$\lim_{n\to\infty} g(a_n) = b$

Apply algebra of limits for sequences.

$$\Rightarrow \lim_{n\to\infty} (f+g)(a_n) = \lim_{n\to\infty} f(a_n) + \lim_{n\to\infty} g(a_n) = a+b$$

$\blacksquare$

Proposition (Squeeze thm for functions)

If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \ell$ and

$\exists$ punctured neighborhood $(x_0-\delta, x_0+\delta)\setminus x_0$ of x_0 s.t.

$$f(x) \leq h(x) \leq g(x) \quad \forall x \in (x_0-\delta, x_0+\delta)\setminus\{x_0\}$$

Then $\lim_{x\to x_0} h(x) = \ell$.

In particular it exists.

Infinite limits

- 2nd definition of limits makes sense

for $x_0 = \pm\infty$ or $\ell = \pm\infty$.

E.g. We say that $\lim_{x \to \pm\infty} f(x) = \ell$

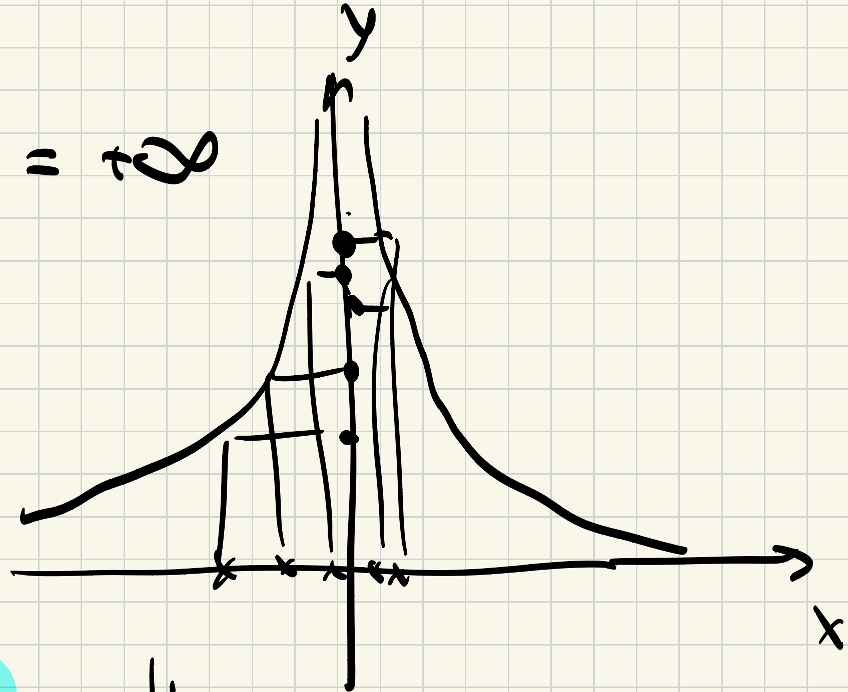
if $\forall$ sequence (a_n) s.t. $\lim_{n \to \infty} a_n = \pm\infty$

we have $\lim_{n \to \infty} f(a_n) = \ell$

- We say that $\lim_{x \to x_0} f(x) = \pm\infty$ if for any (a_n) s.t. $a_n \neq x_0$ and $\lim_{n\to\infty} a_n = x_0$ we have $\lim_{n\to\infty} f(a_n) = \pm\infty$.

Example

$$\lim_{x \to 0} \frac{1}{x^2} = +\infty$$



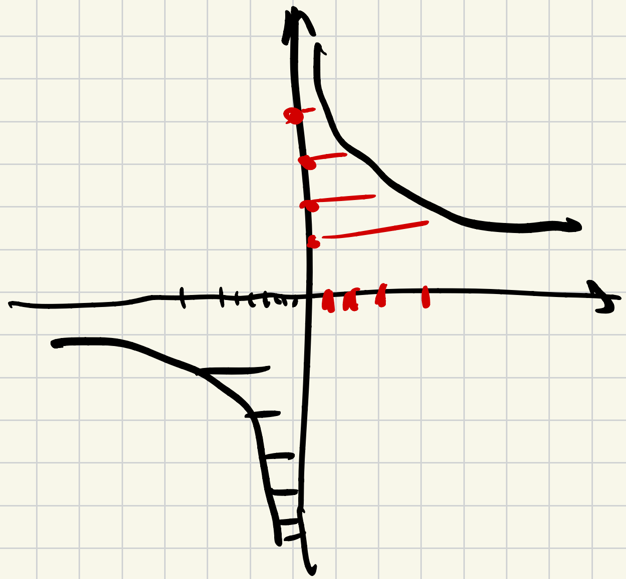
For any (a_n) s.t. $a_n \neq 0$, $\lim_{n\to\infty} a_n = 0$

We get $f(a_n) = \frac{1}{a_n^2}$ so

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \frac{1}{a_n^2} = +\infty$$ since $a_n^2 > 0$, $a_n^2 \to 0$

Example

$$\lim_{x\to 0} \frac{1}{x} \text{ doesn't exist}$$



Consider

- $a_n = \frac{1}{n}$ $\qquad \lim a_n = 0$

- $b_n = -\frac{1}{n}$ $\qquad \frac{1}{\frac{1}{n}} = n$ $\qquad \lim b_n = 0$

But

$$\lim_{n\to\infty} \overbrace{f(a_n)}^{''} = \lim_{n\to\infty} n = +\infty$$

$$\neq$$

$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} -n = -\infty$$

Algebra of infinite limits

Slogan:

All rules for sequences can be translated to functions.

Example:

Recall: $\lim_{n\to\infty} (x_n) = +\infty$ and

(y_n) is bounded below

$$\Rightarrow \lim_{n\to\infty} (x_n + y_n) = +\infty$$

sequence statement

Let $\lim_{x\to x_0} f(x) = +\infty$ let $g(x)$ be bounded below in some punctured neighborhood of x_0 then $\lim_{x\to x_0} (f+g) = +\infty$

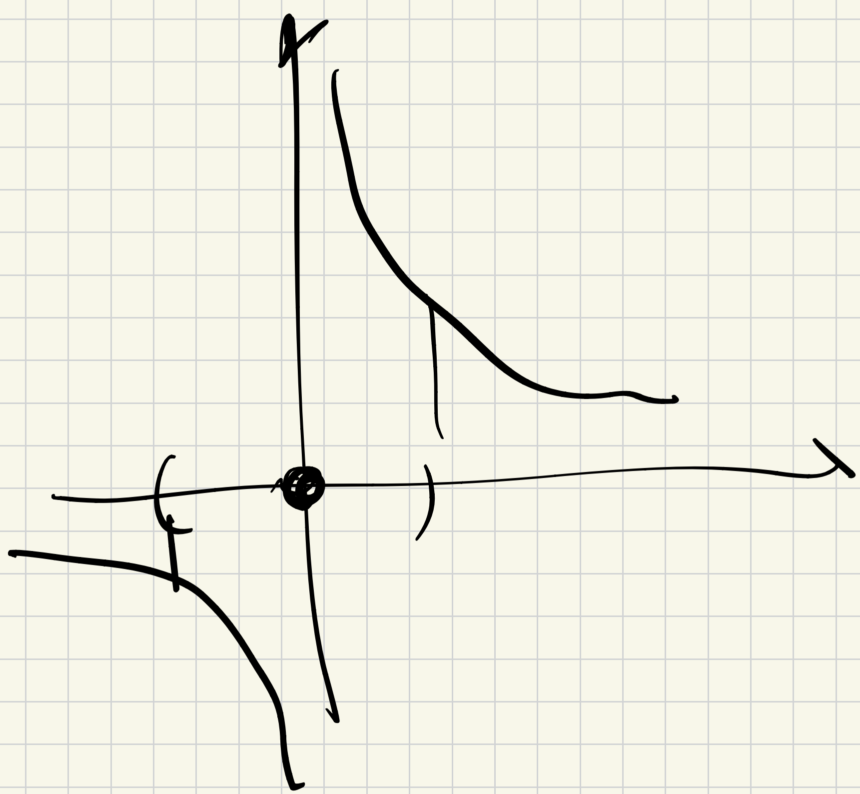
function statement

More statements are

in Prop 1.36 of

Notes on Functions

E.g $\frac{1}{x}$ is not bounded in any punctured neighbourhood of 0.



$\underline{\text{E.g}}$ $\boxed{\lim_{x \to 0} \left(\frac{1}{x^2} + \cos(x) \right)}$

Note that $\lim_{x \to 0} \frac{1}{x^2} = +\infty$

Also $\cos(x)$ is bounded below.

$\Rightarrow \lim_{x \to 0} \left(\frac{1}{x^2} + \cos x \right) = +\infty$.

Composition of functions

Definition Let $f: D \to \mathbb{R}$, $g: E \to \mathbb{R}$ be functions s.t. $\operatorname{Im}(f) \subset E$.

Then we define $g \circ f : D \to \mathbb{R}$ as

$$\underbrace{(g \circ f)}_{\text{order is important!}}(x) = g\big(f(x)\big) \quad \forall x \in D.$$

example

$$f(x) = x^2 + 1$$

$$g(x) = x^3 - x$$

$$\Rightarrow \quad g \circ f(x) = g\big(f(x)\big) =$$

$$= g\big(\underbrace{x^2 + 1}\big) = (x^2+1)^3 - (x^2+1)$$

Remark $$f \circ g \neq g \circ f \quad !!!$$

Example $$f = x^2$$

$$g = \cos(x)$$

then $$g \circ f = \cos(x^2)$$

$$\neq$$

$$f \circ g = (\cos(x))^2 = \cos^2(x)$$

Composition and limits

Prop Assume $f: D \to \mathbb{R}$, $g: E \to \mathbb{R}$ are functions s.t.

1) $Im(f) \subset E$ ← Need to get $g \circ f$ to be defined

2) f is continuous at x_0

3) g is continuous at $y_0 = f(x_0)$

Then $g \circ f$ is continuous at x_0

Composition of continuous functions is continuous.

<u>Example</u> $h(x) = \cos^2 x - 3\cos x$

We can write $h = g \circ f$ for

$f(x) = \cos(x)$ and $g(x) = x^2 - 3x$

- $\cos(x)$ is continuous at 0.
- Any polynomial function is continuous for every $x \in \mathbb{R}$.

$\Rightarrow$

$\Rightarrow \cos^2 x - 3\cos x$ is continuous at 0.

Showed today

In fact $\cos(x)$ continuous everywhere.

Proposition Let $f: D \to \mathbb{R}$, $g: E \to \mathbb{R}$ be two functions s.t.:

1) $\mathrm{Im}(f) \subset E$ for $g \circ f$ to be well defined

2) $\lim_{x \to x_0} f(x) = y_0$ 3) $\lim_{y \to y_0} g(y) = \ell$

4) $\exists$ a punctured neighborhood of x_0 s.t. for every x in it $f(x) \neq y_0$ } technical condition

Then $\lim_{x \to x_0} (g \circ f)(x) = \ell$

Example

$$\lim_{x \to 0} \underbrace{\frac{\sin(x^2)}{x^2}}_{h(x)} = ?$$

Note that $h(x) = g \circ f$ with

$$f(x) = x^2 \quad , \quad g(x) = \frac{\sin(x)}{x}$$

$$\lim_{x \to \underset{x_0}{\overset{\shortparallel}{0}}} x^2 = \underset{y_0}{\overset{\shortparallel}{0}} \qquad \lim_{y \to y_0} \frac{\sin(x)}{x} = 1$$

By squeeze thm.

So we get that

$$\lim_{x \to 0} g \circ f = \lim_{y \to 0} g = 1 .$$

$$= \lim_{x \to 0} f$$